UNIT-4

TRANSIENTS AND RESONANCE IN RLC CIRCUIT

Here we will examine two types of simple circuits:

→ A circuit comprising a resistor and capacitor

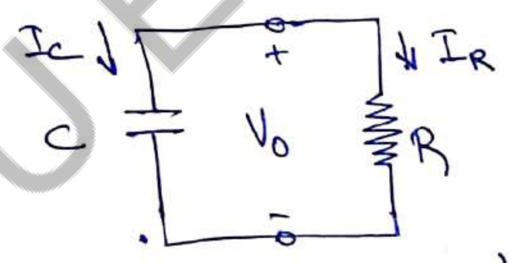
→ and a circuit comprising a resistor and an inductor.

These are called RC and RL.

The Source-free RC Circuit:

* A source-free RC circuit occurs when its DC source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

→ Consider a series combination of a resistor and an initially charged capacitor.



→ Objective is to determine the circuit response, assume to be the voltage $v(t)$ across the capacitor.

Since capacitor initially charged, we can assume that at time $t=0$, the initial voltage is,

$$v(0) = V_0.$$

When the Corresponding Value of energy Stored

$$w(0) = \frac{1}{2} C V_0^2$$

Applying KCL at the node of the Circuit,

$$i_C + i_R = 0$$

We know that $i_C = C\frac{dv}{dt}$ and $i_R = \frac{v}{R}$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

by rearranging

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both Sides,

$$\boxed{\frac{dx}{x} = \ln x}$$

$$\ln v = -\frac{t}{RC} + \ln A$$

$A \rightarrow$ integration Constant, thus,

$$\frac{\ln v}{\ln A} = -\frac{t}{RC}$$

Taking Powers of e produce

$$\boxed{e^{\ln x} = x}$$

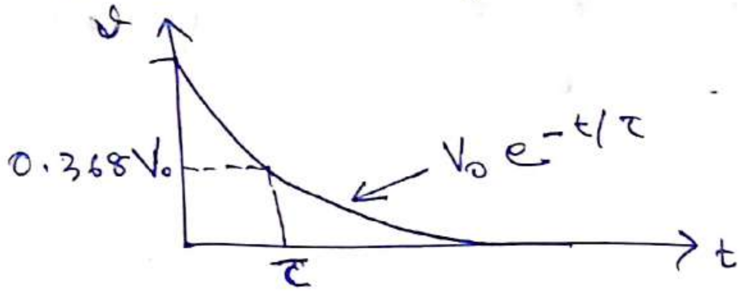
$$\frac{v}{A} = e^{-t/RC}$$

$$\therefore v(t) = Ae^{-t/RC}$$

at initial Condition $v(0) = A = V_0$

$$\boxed{v(t) = V_0 e^{-t/RC}}$$

This shows the voltage response of the RC circuit is an exponential decay of the initial voltage.



The response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the natural response of the circuit.

from the graph, As t increases the voltage decreases toward zero.

The rapidity with which the voltage decreases is expressed in terms of the time constant (τ).

that, $t = \tau$

then $\boxed{V(t) = V_0 e^{-t/RC}}$

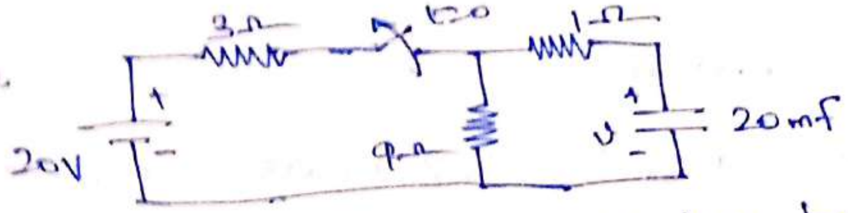
where $\tau = RC$

$$\boxed{V(t) = V_0 e^{-t/\tau}} = V_0 e^{-1}$$

$$= 0.368 V_0$$

In terms of time constant

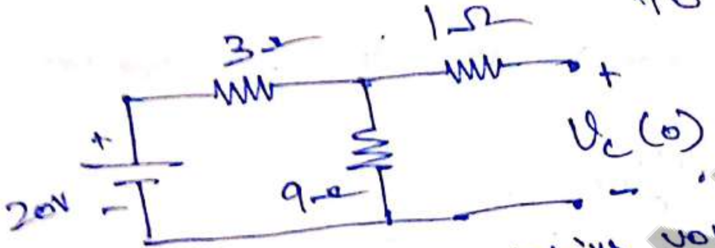
$$\boxed{v(t) = V_0 e^{-t/\tau}}$$



The switch in the circuit has been closed for a long time, and its opened at $t=0$. Find $v(t)$ for $t \geq 0$. Calculate the intial energy stored in the capacitor.

Solution:

for $t<0 \rightarrow$ Switch closed, the capacitor is an open circuit, to DC.



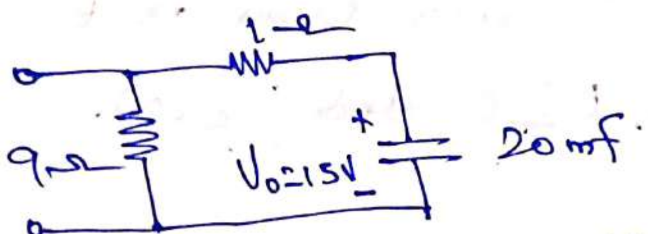
using voltage division Rule

$$v_c(t) = 20 \times \frac{9}{9+3} = 15\,V \quad t<0$$

Voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t=0^-$ is same at $t=0$

$$v_c(0) = V_0 = 15\,V.$$

for $t>0$, the switch is opened,



Source free RC circuit.

$R_{eq} = 1+9 = 10\,\Omega$

time constant,

$$\tau = R_{eq} C = 10 \times 20 \times 10^{-3} = 0.2\ s$$

the voltage across the capacitor for $t \geq 0$ is

$$v(t) = V_c(0)\, e^{-t/\tau}$$

$$= 15\, e^{-t/0.2}\ V$$

$$\boxed{v(t) = 15\, e^{-5t}\ V}$$

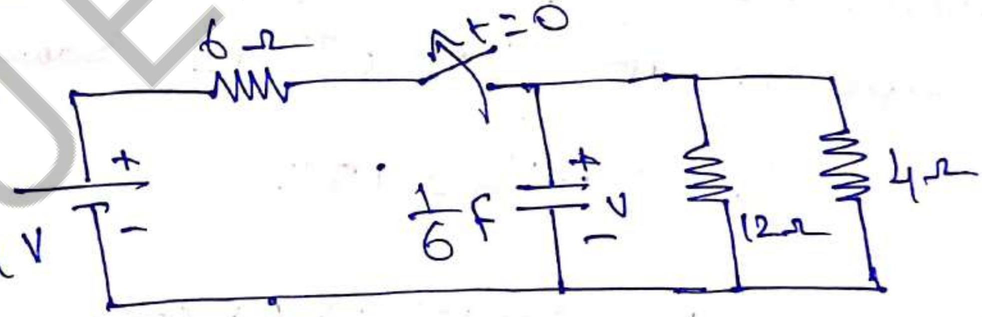
Initial energy stored in the capacitor is

$$W_c(0) = \frac{1}{2} C\, v_c^2(0)$$

$$= \frac{1}{2} \times 20 \times 10^{-3} \times 15^2$$

$$\boxed{W_c(0) = 2.25\ J}$$

Ex:-4:

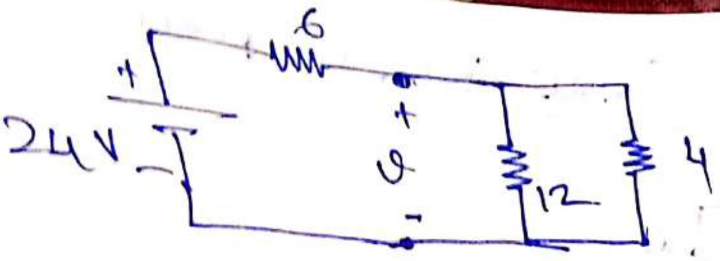


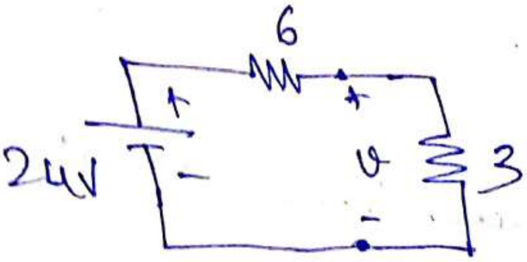
If the switch opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $W_c(0)$.

Solution:

for $t < 0 \rightarrow$ switch closed

capacitor as open circuited.





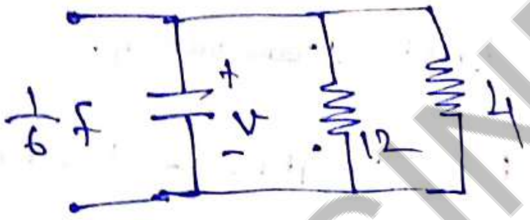
$\Rightarrow 24 \times \dfrac{3}{6+3}$

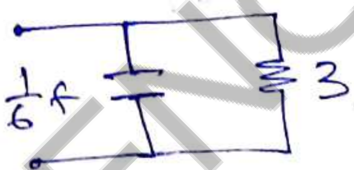
$= 8\ V$

$v_c(t) = 8\ V \quad t < 0$

for $t \geq 0$ Switch opened

$v_c(0) = V_0 = 8\ V$





$R_{eq} = 3$

$\tau = RC = 3 \times \frac{1}{6} = 0.5\ s$

Voltage across capacitor $t \geq 0$

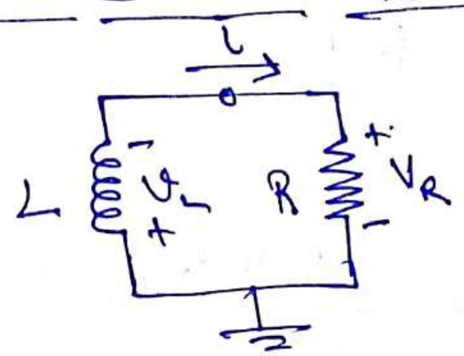
$v(t) = v_c(0)\, e^{-t/0.5}$

$v(t) = 8 e^{-2t}\ V$

$w_c(0) = \frac{1}{2} C V_c^2(0) = \frac{1}{2} \times \frac{1}{6} \times 8^2$

$= 5.33\ J$

THE SOURCE FREE RL CIRCUIT



Our goal to determine the circuit Response.

We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously.

Assume at $t=0$ the inductor has initial current I_0,

$$i(0) = I_0.$$

The corresponding energy stored in the inductor,

$$w(0) = \frac{1}{2} L I_0^2 .$$

Apply KVL around the loop,

$$V_L + V_R = 0$$

but, $V_L = L\frac{di}{dt}$ & $V_R = iR$, thus

$$L\frac{di}{dt} + Ri = 0$$

for Simplification, divide by 'L'

$$\frac{di}{dt} + \frac{R}{L} i = 0$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big]_{I_0}^{i(t)} = -\frac{Rt}{L}\Big]_0^t$$

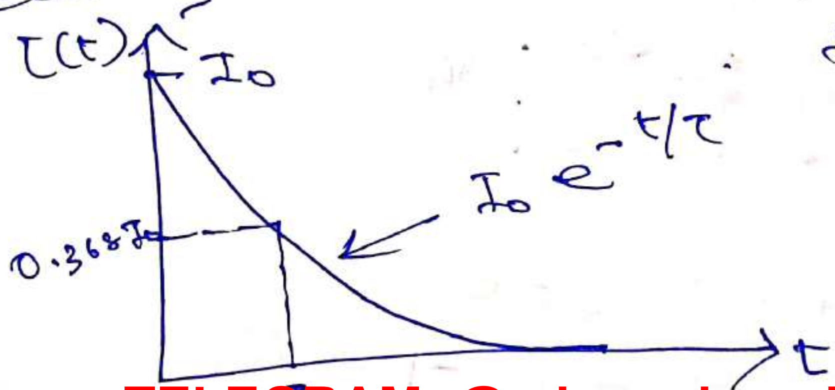
$$\ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

Taking e powers

$$i(t) = I_0 e^{-Rt/L}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current.



time constant for RL

$$\tau = \frac{L}{R}$$

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

then we can find voltage across the resistor as

$$\boxed{V_R(t) = iR = I_0 R e^{-t/\tau}}$$

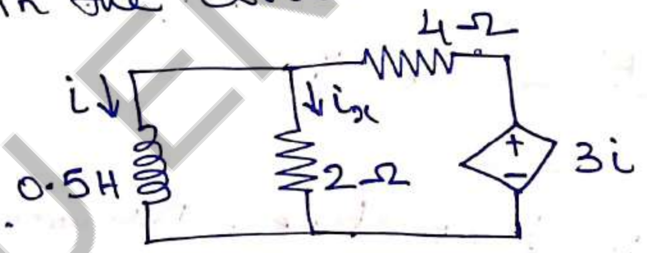
Power dissipate in the Resistor

$$\boxed{P = V_R i = I_0^2 R e^{-2t/\tau}}$$

$$\left[= I_0 R e^{-t/\tau} \times I_0 e^{-t/\tau} = I_0^2 R e^{-2t/\tau} \right]$$

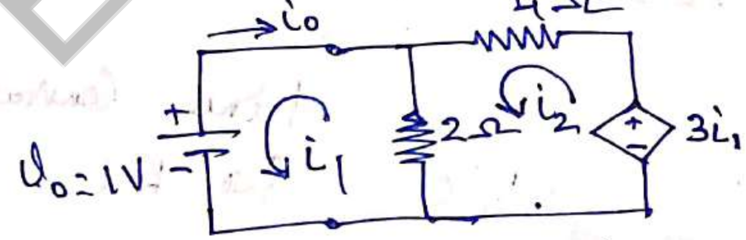
Problem:

Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$ in the circuit.



Solution:

Because of the dependent source, we insert a voltage source with $v_0 = 1V$,



Apply KVL:

$$1 + 2(i_1 - i_2) = 0$$

$$2i_1 - 2i_2 = -1 \quad (1)$$

$4i_2 + 2(i_2 - i_1) - 3i_1 = 0$

$4i_2 + 2i_2 - 2i_1 - 3i_1 = 0$

$6i_2 - 5i_1 = 0 \quad (2)$

$i_2 = \frac{5}{6} i_1$

Sub i_2 value in (1)

$2i_1 - 2\left(\frac{5}{6}\right)i_1 = -1$

$i_1\left[2 - \frac{10}{6}\right] + 1 = 0$

$i_1\left[\frac{2}{6}\right] + 1 = 0$

$2i_1 - 2i_2 = -1$

$-5i_1 + 6i_2 = 0$

$\Delta = \begin{vmatrix} 2 & -2 \\ -5 & 6 \end{vmatrix} = 12 - 10 = 2$

$\Delta_1 = \begin{vmatrix} -1 & -2 \\ 0 & 6 \end{vmatrix} = -6$

$i_1 = \frac{\Delta_1}{\Delta} = \frac{-6}{2} = -3\,A$

$\therefore i_0 = -i_1 = -(-3) = \underline{3\,A}$

$\therefore R_{eq} = R_{th} = \frac{V_0}{I_0} = \frac{1}{3}\,\Omega$

Then the time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{0.5}{1/3} = 1.5\,s$$

The current through the inductor is,

$$i(t) = i(0)\,e^{-t/\tau} = \underline{\underline{10\,e^{-t/1.5}}}\ A \quad t > 0$$

Voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5\,\frac{d\,10e^{-t/1.5}}{dt}$$

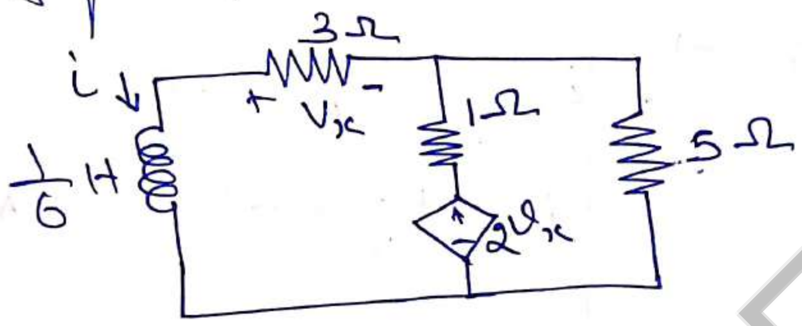
$$= (0.5)(10)\left(-\frac{1}{1.5}\right)e^{-t/1.5}$$

$$v = -3.33\,e^{-t/1.5}\ V$$

Since inductor and 2 Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = \frac{-3.33\, e^{-t/1.5}}{2} = -1.666\, e^{-t/1.5}\ \text{A} \quad t>0.$$

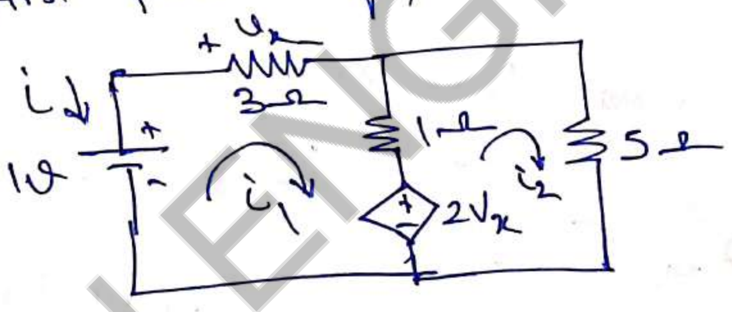
Ex 2: find i and V_x in the circuit in figure, Let $i(0) = 5$ A



Solution: Given: $i(0) = 5$

find : i and V_x

first find R_{eq},



Apply KVL

$$-1 + 3i_1 + 1(i_1 - i_2) + 2V_x = 0$$

$$4i_1 - i_2 + 2(3i_1) = -1$$

$$10i_1 - i_2 = +1 \quad \text{—} \ (1)$$

Loop 2

$$1[i_2 - i_1] + 5i_2 - 2V_x = 0$$

$$i_2 - i_1 + 5i_2 - 2(3i_1) = 0$$

$$6i_2 - 7i_1 = 0$$

$$-7i_1 + 6i_2 = 0 \quad \text{—} \ (2)$$

$$10i_1 - i_2 = +1$$
$$-7i_1 + 6i_2 = 0$$

$$\Delta = \begin{vmatrix} 10 & -1 \\ -7 & 6 \end{vmatrix} = 60 - 7 = 53$$

$$\Delta_1 = \begin{vmatrix} +1 & -1 \\ 0 & 6 \end{vmatrix} = +6$$

$$i_1 = \frac{+6}{53} = +0.113$$

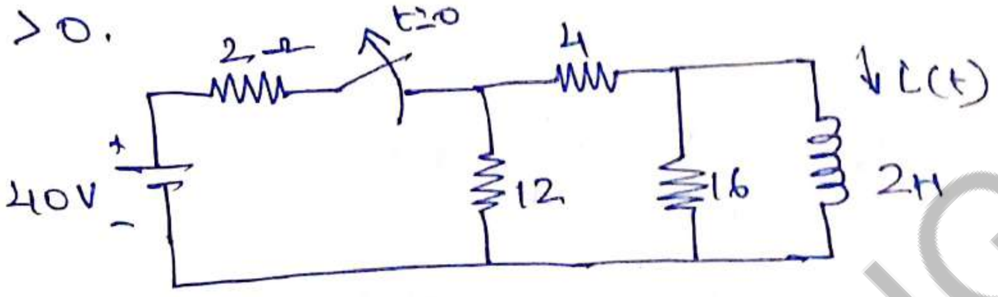
$$R_{eq} = R_{th} = \frac{V_0}{I_0} = \frac{1}{+0.113} = 8.8\,\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{1/6}{8.8} = 0.0189$$

$$i(t) = i(0)\,e^{-t/\tau}$$
$$i(t) = 5\,e^{-t/0.0189} = \underline{\underline{5\,e^{-53t}}}\ A$$

$$V_x = 3 \times i(t)$$
$$= 3 \times [-5e^{-53t}]$$
$$V_x = -15\,e^{-53t}\ V$$

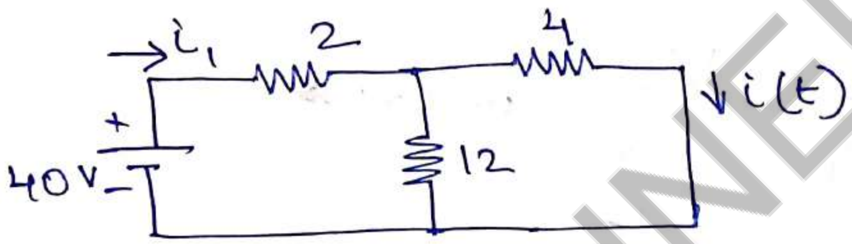
Ex 3: The Switch in the circuit has been closed for a long time. At $t=0$ the Switch is Opened. Calculate $i(t)$ for $t>0$.

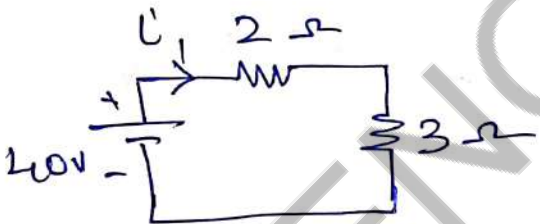


Solution:

at $t<0$ → Switch closed, Inductor act as Short circuit.

→ 16 Ω Resistor is Short circuited.





$$i_1 = \frac{40}{5} = 8\,A$$

and $i(t)$ obtain by current divison rule

$$i(t) = 8 \times \frac{12}{12+4} = 6A \quad t<0$$

Since current through an inductor cannot change instantaneously.

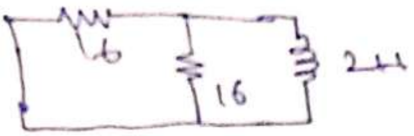
$$i(0) = i(0^-) = 6A \quad \text{at } t=0$$

When $t>0$ → Switch Opened,

voltage source dis connected.

Source free RL circuit

Req ⇒



$$= \frac{16 \times 16}{16+16} = 8\,\Omega$$

∴ time constant

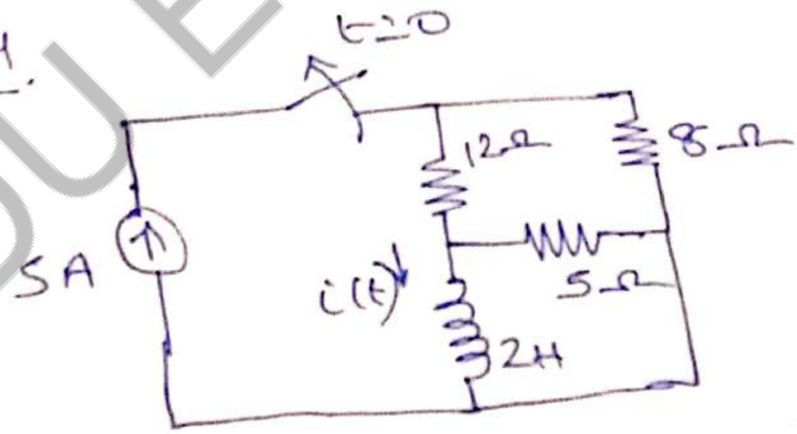
$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\,s$$

Thus

$$i(t) = i(0)\,e^{-t/\tau}$$

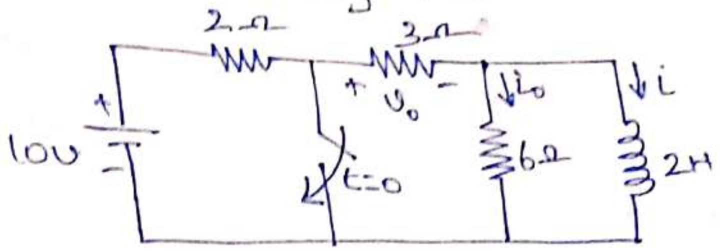
$$\underline{\underline{i(t) = 6\,e^{-4t}\,A}} \quad t > 0$$

Ex: 4



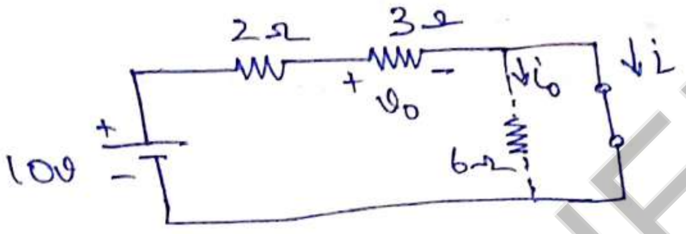
Ans: $i(t) = \underline{\underline{2\,e^{-2t}\,A}}$

Ex: 7.5 In the circuit shown, find i_o, v_o & i for all time, assuming that the switch was open for a long time.



Solution:

⇒ for $t < 0$ → Switch is open, then DC source connected to Inductor, so it behaves like short circuit.



$$i_o = 0 \; ; \; t < 0$$

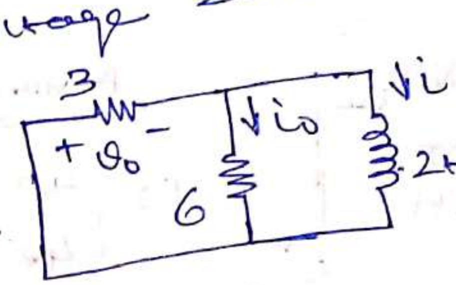
$$i = \frac{10}{3+2} = 2A \; ; \; t < 0$$

$$v_o = i \times 3 = 2 \times 3 = 6\,V \; ; \; t < 0$$

then $\underline{i(0) = 2A}$.

⇒ for $t > 0$ → Switch is closed, so that voltage source is short circuited.

$$i(t) = i(0)\, e^{-t/\tau}$$



$$R_{eq} = \frac{3 \times 6}{3+6} = \frac{18}{9} = 2\,\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{2}{2} = 1 \text{ Sec}$$

$$i(t) = 2e^{-t/1}\ A$$

$$\therefore\ i(t) = 2e^{-t}\ A \quad \text{at } t \geq 0$$

Need to find i_0 i.e. Current through $6\,\Omega$ Resistor

by Current division Rule,

$$i_0 \text{ or } i_6 = (-2e^{-t}) \times \frac{3}{9}$$

[Note: $i(t)$ is reverse direction when current through in $6\,\Omega$]

$$i_0(t) = -\frac{2}{3}e^{-t}\ A \quad \text{at } t > 0$$

$$v_0(t) = v_L = L\frac{di}{dt}$$

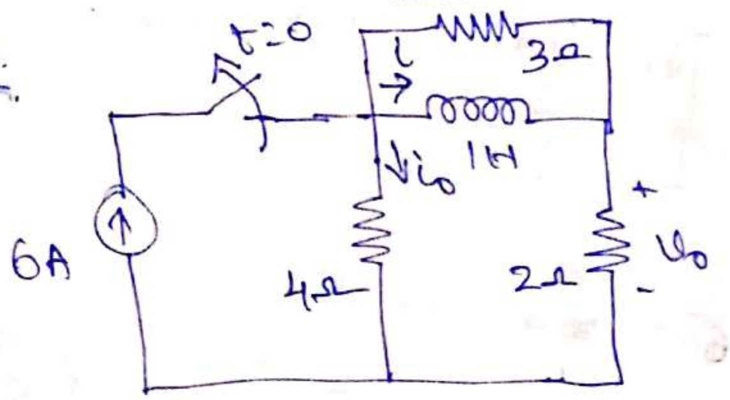
$$= 2 \times \frac{d(-2e^{-t})}{dt}$$

$$= 2 \times [-2(-1)e^{-t}]$$

$$= 2 \times 2e^{-t}$$

$$v_0(t) = 4e^{-t}\ V$$

Ex:



Answers

$$i: \begin{cases} 4A & t<0 \\ 4e^{-2t}\ A & t \geq 0 \end{cases}$$

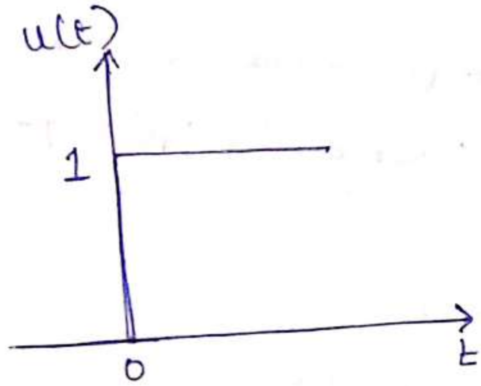
$$v_0 = \begin{cases} 4V & t<0 \\ -8/3\, e^{-2t}\ V & t>0 \end{cases}$$

$$i_0 = \begin{cases} 2A & t<0 \\ -4/3\, e^{-2t}\ A & t>0 \end{cases}$$

Unit Step function:

The Unit Step function $u(t)$ is '0' for Negative values of 't' and '1' for positive Value of 't'.

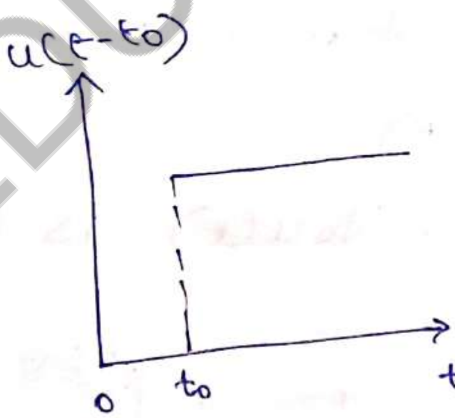
i.e. $$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



The Unit Step function is Undefined at $t=0$, where it changes abruptly from 0 to 1.

→ If the abrupt changes Occurs at $t = t_0$, then the Unit Step function becomes,

$$u(t-t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

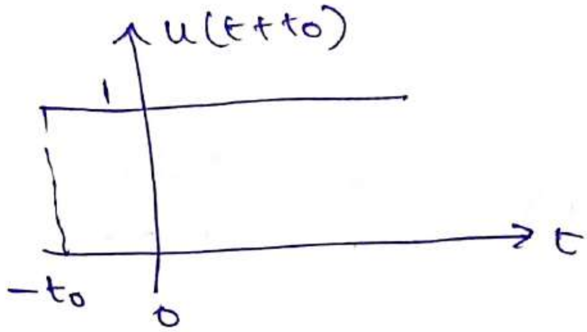


that $u(t)$ delayed by t_0 seconds.

→ If the change is at $t = -t_0$, then the Unit Step function becomes,

$$u(t+t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

i.e $u(t)$ is advanced by t_0 seconds.



We use the step function to represent an abrupt change in voltage or current.

examp, the voltage,

$$v(t) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

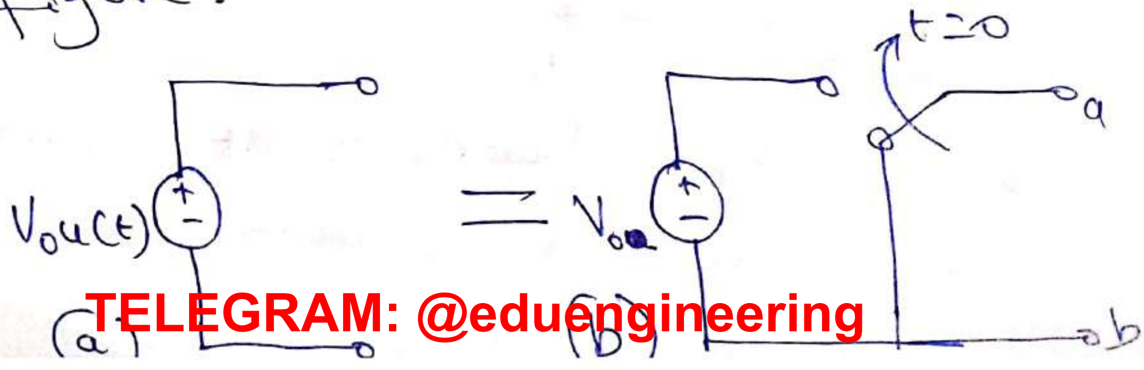
it expresses in terms of unit step function as,

$$v(t) = V_0 u(t-t_0)$$

If $t_0 = 0$, then, step voltage,

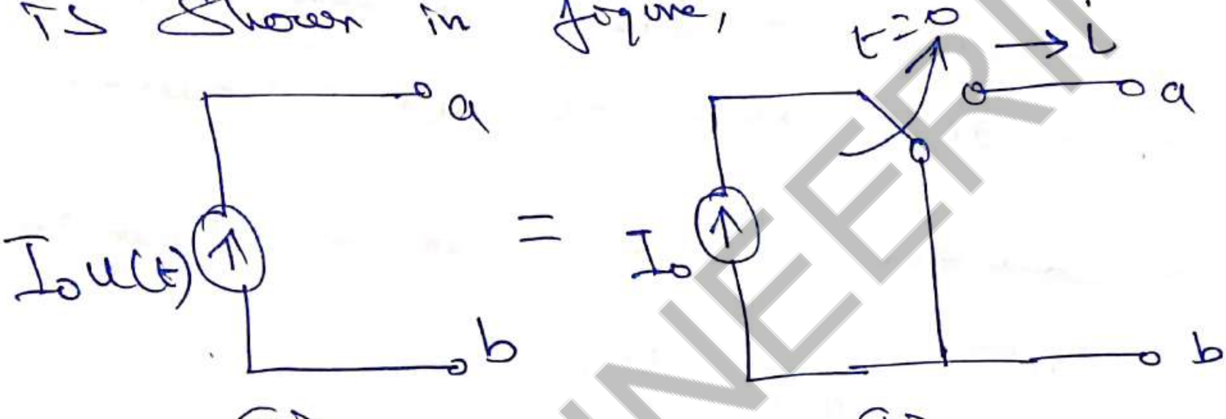
$$v(t) = V_0 u(t).$$

A voltage source $V_0 u(t)$ is shown in figure.



from the figure(b) it is evident that terminals a-b are short circuited $v=0$ for $t<0$, and $v=V_0$ at terminals for $t>0$.

Similarly, a current source of $I_0 u(t)$ is shown in figure,



(a) (b)

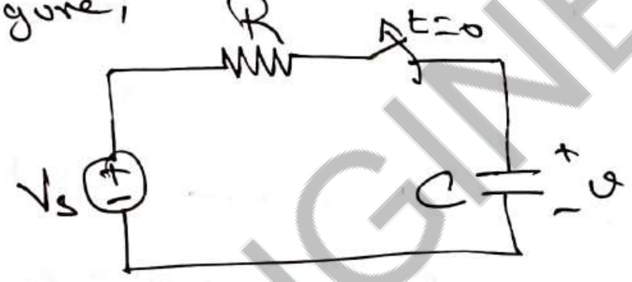
from figure (b), Noticed that terminal a & b open circuited $i=0$ for $t<0$, and $i=I_0$ flow for $t>0$.

STEP RESPONSE OF AN RC CIRCUIT

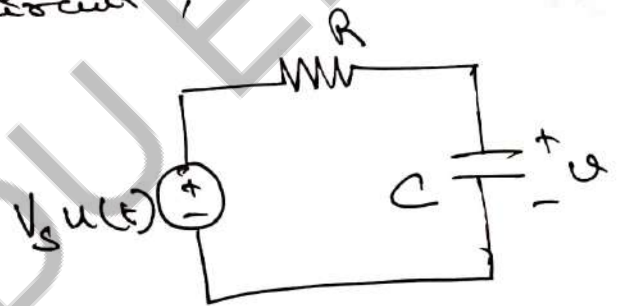
When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function and the response is known as a step response.

→ The step response is the response of the circuit due to sudden application of a dc voltage or current source.

→ Consider the RC circuit shown in figure,



which can be replaced by the circuit,



V_s → is a constant dc voltage source

Assume an initial voltage V_0 on the capacitor, the voltage on the capacitor cannot change instantaneously,

$$V(0^-) = V(0^+) = V_0$$

Apply KCL,

$$i_R + i_C = i_0$$

$$\frac{v}{R} + C\frac{dv}{dt} = \frac{V_s u(t)}{R}$$

$$\frac{v}{R} - \frac{V_s u(t)}{R} + C\frac{dv}{dt} = 0$$

$$\frac{v - V_s u(t)}{R} + C\frac{dv}{dt} = 0$$

divide by 'C'

$$\frac{v - V_s u(t)}{RC} + \frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s u(t)}{RC}$$

where v is voltage across capacitor at $t > 0$. So $u(t) = 1$ at $t > 0$.

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

Rearranging terms,

$$\frac{dv}{dt} = -\frac{(v - V_s)}{RC}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrating both sides

$$\int^{v(t)} \frac{dv}{v - V_s} = \int^{t} -\frac{dt}{RC}$$

$$\ln(v - V_s)\Big]_{V_0}^{v(t)} = -\frac{t}{RC}\Big]_0^t$$

$$\ln[v(t) - V_s] - \ln[V_0 - V_s] = -\frac{t}{RC} + 0$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking exponential of both sides,

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/RC}$$

where $\tau = RC$

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0.$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

This is known as Complete Response of the RC circuit to a sudden application of DC voltage source. Capacitor is initially charged.

If we assume capacitor is uncharged initially $V_0 = 0$, so the equation becomes,

$$v(t) = \begin{cases} 0 & , t < 0 \\ V_s(1 - e^{-t/\tau}) & , t > 0 \end{cases}$$

Can be written alternatively as,

$$\boxed{v(t) = V_s(1 - e^{-t/\tau})\,u(t)}$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged.

→ The current through the capacitor is obtained from above equation, by using

$i(t) = C\,dv/dt$

$$i(t) = C\frac{dv}{dt} = C\frac{d}{dt}V_s(1 - e^{-t/\tau})\,u(t)$$

$$\tau = RC \qquad t > 0, \quad u(t) = 1$$

$$i(t) = (C)(V_s)\left[-\left(-\frac{1}{\tau}\right)e^{-t/\tau}\right]$$

$$i(t) = \frac{C}{\tau}V_s e^{-t/\tau}$$

$$\boxed{i(t) = \frac{V_s}{R}e^{-t/\tau}\,u(t)}$$

$\rightarrow V(t)$

Complete Response = Natural response + forced Response
(Stored energy) (independent Source)

Complete Response = Transient Response + Steady State Response
(temporary part) (permanent part)

Natural Response = Transient Response
forced Response = Steady State Response

A Complete Response may be written as,

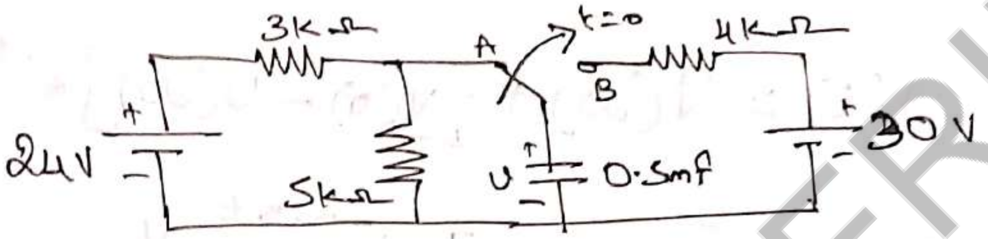
$$\boxed{v(t) = V(\infty) + [V(0) - v(\infty)]e^{-t/\tau}}$$

$v(0) \rightarrow$ initial Voltage
$v(\infty) \rightarrow$ final or Steady State value.

To find Step Response of an RC Circuit Requires three things,

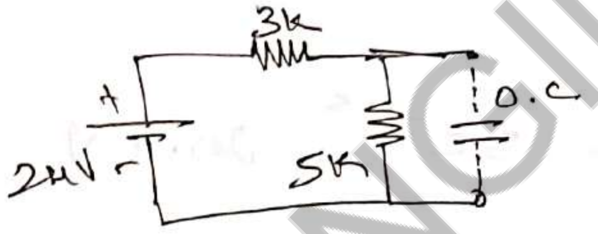
1. The initial Capacitor Voltage $v(0) \ldots t<0$
2. The final Capacitor Voltage $V(\infty)$ } $t>0$
3. The time constant τ.

Ex: 1. The Switch in figure has been in position A for a long time. At $t=0$, the Switch moves to B. Determine $v(t)$ for $t>0$ and Calculate its value at $t=1s$ and $4s$.



Solution:

(*) for $t<0 \rightarrow$ Switch is at position 'A'. Capacitor acts like open circuit.



Voltage across 5k is voltage across capacitor.

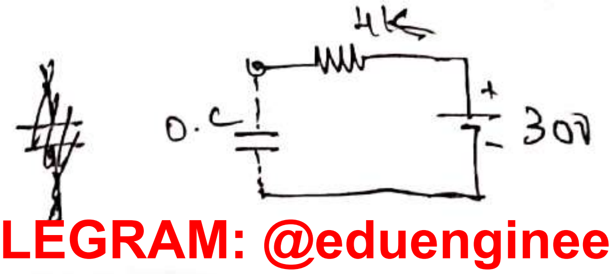
Using voltage division Rule

$$v(0^-) = 24 \times \frac{5}{3+5} = 15V$$

(*) The capacitor voltage cannot change instantaneously, so,

$$v(0) = v(0^-) = v(0^+) = 15V$$

(*) for $t>0 \rightarrow$ Switch is in position 'B'.



$R_{th} = 4k\Omega$

$\wedge$ $\quad$ $-v(t)$

time Constant,

$$\tau = R_{th} C = 4\times10^{3} \times 0.5\times10^{-3} = 2\,s$$

→ the capacitor acts like Open Circuit, to dc at Steady State,

$$V(\infty) = 30V. \quad t>0$$

Thus,

$$v(t) = V(\infty) + [v(0) - v(\infty)]\, e^{-t/\tau}$$

$$= 30 + [15-30]\, e^{-0.5t} \; V$$

$$v(t) = 30 - 15e^{-0.5t} \; V$$

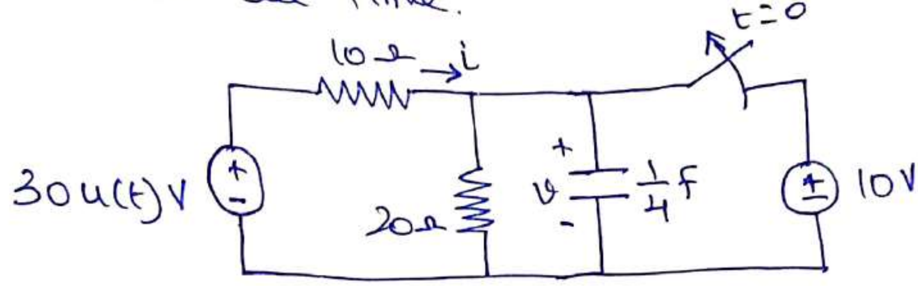
At $t=1$

$$v(1) = 30 - 15e^{-0.5} = 20.9\, V$$

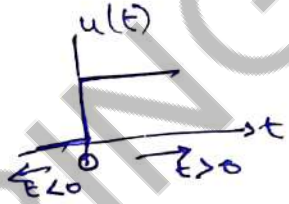
At $t=4$,

$$v(4) = 30 - 15e^{-2} = 27.97\, V$$

Ex: 2: In figure the Switch has been closed for a long time and is Opened at $t=0$. find i and v for all time.



Solution:



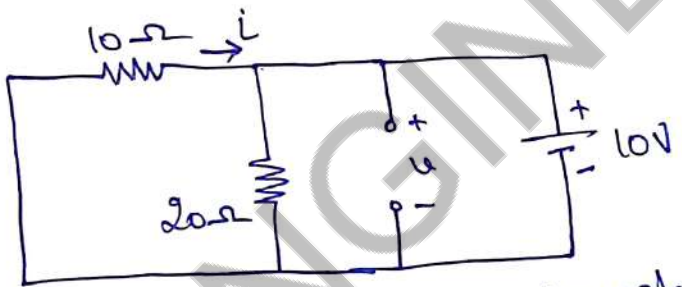
for $t<0 \rightarrow$ Switch Closed

at $t<0$ the $30u(t) = 0$

So 10 V source connected with capacitor

It behaves Open Circuited.

$$30u(t) = \begin{cases} 0 & t<0 \\ 30 & t>0 \end{cases}$$



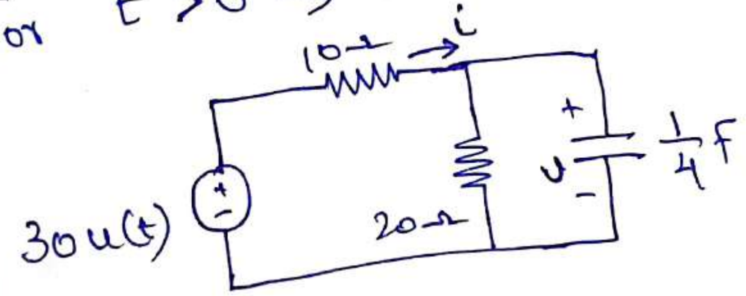
No current flow through 20Ω Resistor,

$$i = -\frac{v}{10} = -\frac{10}{10} = -1\ A \quad ; \quad v = 10\ V$$

Capacitor voltage cannot change instantaneously

So, $v(0) = v(0^-) = 10V$

for $t>0 \rightarrow$ Switch Opened.



After a long time, the circuit reaches Steady State & Capacitor acts Like an Open Circuit again.

Steady State voltage, $v(\infty)$ obtain using voltage division,

$$v(\infty) = 30 \times \frac{20}{20+10} = 20\,V$$

find R_{th} by,

$$R_{th} = \frac{10 \times 20}{10+20} = \frac{200}{30} = \frac{20}{3}\,\Omega$$

time Constant,

$$\tau = R_{th} C = \frac{20}{3} \times \frac{1}{4} = \frac{20}{12} = \frac{5}{3}\,s$$

Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]\, e^{-t/\tau}$$

$$= 20 + [10 - 20]\, e^{-(3/5)t}$$

$$v(t) = (20 - 10\, e^{-0.6t})\,V$$

To Obtain i,

$i \rightarrow$ Sum of Current through $20\,\Omega$ and Capacitor.

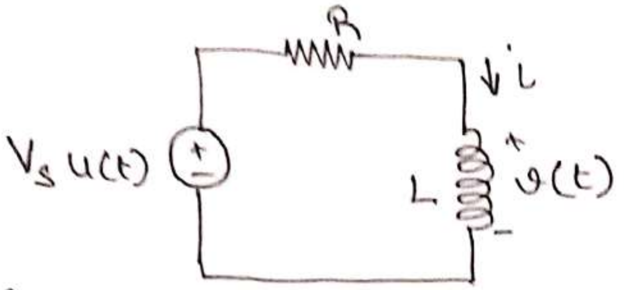
$$i = \frac{v}{20} + C\frac{dv}{dt}$$

$$= \frac{20 - 10\, e^{-0.6t}}{20} + \frac{1}{4} \times \frac{d[20 - 10\, e^{-0.6t}]}{dt}$$

$$= (1 + e^{-0.6t})\,A$$

STEP RESPONSE OF AN RL CIRCUIT

Consider the RL circuit,



Our goal is to find the inductor current i as the circuit response.

Let the response be the sum of the natural current and the forced current.

$$i = i_n + i_f \quad \text{——} \ (1)$$

$i_n \rightarrow$ Natural response

$i_f \rightarrow$ forced response

We know that natural response is the decaying exponential.

$$i_n = A e^{-t/\tau} \quad \text{——} \ (2)$$

A → is a constant

forced response is the value of the current long time after switch is closed. At the time the inductor becomes short circuit and the voltage across it is zero. The entire source voltage V_s appears across R, thus,

$$i_f = \frac{V_s}{R} \quad \text{——} \ (3)$$

Sub equation (2) & (3) in (1)

$$i = A e^{-t/\tau} + \frac{V_s}{R} \quad \text{——} \ (4)$$

We now determine the constant A from the value of 'i'. Let I_0 be initial current, The current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \quad \text{— (5)}$$

Thus at $t=0$, then Equation (4) becomes,

$$I_0 = A + \frac{V_s}{R}$$

then, $A = I_0 - \frac{V_s}{R}$

Substituting 'A' in equation (4)

$$i(t) = \frac{V_s}{R} + \left[I_0 - \frac{V_s}{R}\right] e^{-t/\tau} \quad \text{— (5)}$$

This is the complete response of the RL circuit.

The Response in equation (5) may be written as,

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau} \quad \text{— (6)}$$

$i(0) \rightarrow$ initial value

$i(\infty) \rightarrow$ final value

To find the step response of an RL circuit requires three things,

1. The initial inductor current $i(0)$ at $t=0^+$
2. final inductor current $i(\infty)$.
3. The time constant τ.

$\left.\right\} t>0$

If switching takes place at time $t=t_0$ instead of $t=0$, equation becomes,

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t<0 \\ \frac{V_s}{R}(1-e^{-t/\tau}), & t>0 \end{cases}$$

$$\therefore i(t) = \frac{V_s}{R}(1-e^{-t/\tau})\,u(t)$$

This is the step response of the RL circuit with no initial inductor current.

The voltage across the inductor is obtained by,

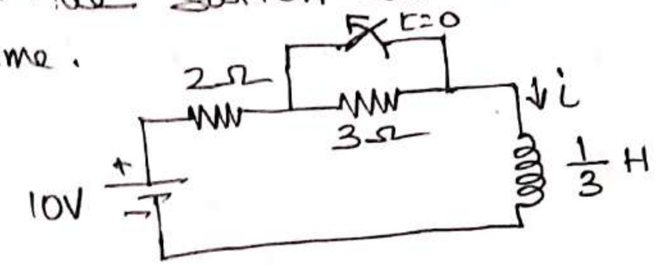
$$v(t) = L\frac{di}{dt} = V_s \frac{L}{\tau R} e^{-t/\tau}$$

where $\tau = \frac{L}{R}$

$$\therefore v(t) = V_s e^{-t/\tau} u(t)$$

Ex 1:

Find $i(t)$ in the circuit for $t>0$. Assume that the switch has been closed for a long time.



Solution:

→ For $t<0$, → the switch is closed position, so, the 3Ω Resistor is short-circuited, & inductor acts like a short circuit.

The current through the inductor at $t=0^-$

$$i(0^-) = \frac{10}{2} = 5A$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^-) = i(0^+) = 5A.$$

→ for $t>0$, the switch is open, 2Ω & 3Ω are in series,

$$i(\infty) = \frac{10}{3+2} = 2A \quad \text{at } t>0.$$

Thevenin Resistance across inductor,

$$R_{th} = 2+3 = 5\Omega.$$

for time constant,

$$\tau = \frac{L}{R_{th}} = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15} \text{ s.}$$

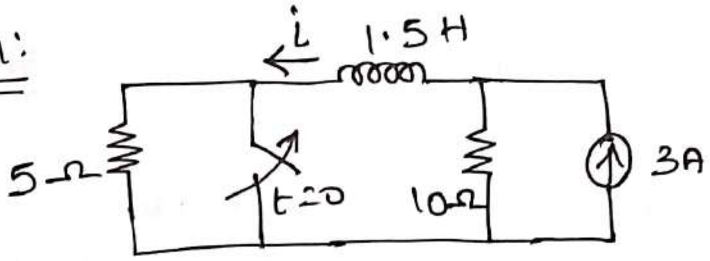
Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$= 2 + (5-2)e^{-15t}$

$i(t) = 2 + 3e^{-15t}$ A $\quad t > 0.$

PRC Prob 1:



The Switch in fig has been closed for a long time. It opens at t=0 find i(t) for t>0.

Solution: for t<0 - Switch closed, so inductor short circuited and 5Ω & 10Ω are short circuited (no current flow through 5Ω & 10Ω).

$i = 3A \quad t < 0.$

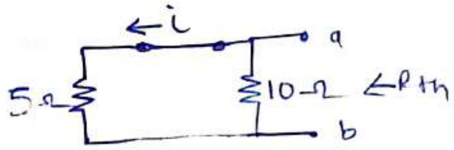
So initial current through inductor cannot change instantaneously,

$i(0) = i(0^-) = 3A \quad t = 0$

for t>0 Switch opened, inductor like short circuit, the current i_1 by CDR

$$i(\infty) = 3 \times \frac{10}{10+5} = 2A \quad t > 0$$

Thevenin Resistance, R_{th},



$$R_{th} = \cancel{\frac{5\times 10}{15}} = \cancel{\frac{50}{15}} = \cancel{3.33\Omega}$$

$$= 5 + 10 = 15\Omega$$

time constant,

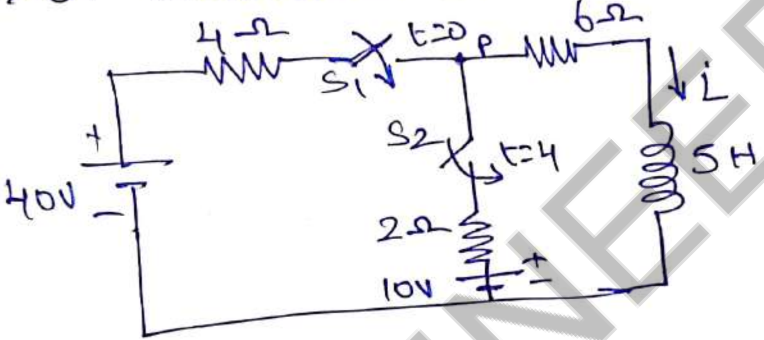
$$\tau = \frac{L}{R_{th}} = \frac{1.5}{\cancel{3.33}\ 15} = \cancel{0.45}\ 0.1\ s$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]\, e^{-t/\tau}$$

$$= 2 + [3 - 2]\, e^{-t/0.1}$$

$$i(t) = (2 + e^{-10t})\,A, \quad t > 0.$$

Ex2: At $t=0$, Switch 1 in fig is closed, and Switch 2 is closed 4 s later. Find $i(t)$ for $t>0$. Calculate i for $t = 2s$ and $t = 5s$



Solution: Need to consider three time intervals, separately.

$t<0$, $0<t<4$, $t>4$

→ for $t<0$ → switches S_1 & S_2 are open, so

$i = 0$. $\quad i(0^-) = i(0) = 0$

→ for $0 < t < 4$ → S_1 is closed so 4Ω & 6Ω are in series.

$$i(\infty) = \frac{40}{4+6} = 4\,A$$

$$R_{th} = 4 + 6 = 10\,\Omega$$

$$\tau = \frac{L}{R_{th}} = \frac{5}{10} = \frac{1}{2} = 0.5\,s$$

Then,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$i(t) = 4 + (0-4)e^{-2t} = 4(1-e^{-2t})\,A$$

$0 < t < 4$

→ for $t > 4$, S_2 is closed, 10 volt source is connected. This sudden change does not affect the inductor current because the current cannot change abruptly. The initial current is,

$$i(4) = i(4^-) = 4(1-e^{-8}) \simeq 4A$$

To find $i(\infty)$, let v be the voltage at node p in fig, using KCL,

$$\frac{40-V}{4} + \frac{10-V}{2} = \frac{V}{6}$$

$$\frac{40}{4} - \frac{V}{4} + \frac{10}{2} - \frac{V}{2} = \frac{V}{6}$$

$$10 + 5 - \frac{V}{4} - \frac{V}{2} - \frac{V}{6} = 0$$

$$-V\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{6}\right] = -15 \Rightarrow v = 16.3636$$

$$i = \frac{v}{6} = \frac{16.3636}{6} = 2.727A$$

$\therefore\ i(\infty) = 2.727A$

Thevenin resistance at inductor terminal,

$$R_{th} = [4 \| 2] + 6 = \frac{4\times 2}{6} + 6 = \frac{22}{3}\,\Omega$$

$$\tau = \frac{L}{R_{th}} = \frac{5}{22/3} = \frac{15}{22} \text{ s}$$

$$\therefore \quad i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau} \qquad t \geq 4$$

[Note: $(t-4)$ is time delay]

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}$$

$$= 2.727 + 1.273\, e^{-(t-4)/0.6818}$$

$$i(t) = 2.727 + 1.273 e^{-1.4667(t-4)}, \quad t \geq 4$$

$$\therefore \quad i(t) = \begin{cases} 0, & t < 0 \\ 4(1-e^{-2t}) & 0 < t < 4 \\ 2.727 + 1.273 e^{-1.466(t-4)}, & t > 4 \end{cases}$$

At: $t = 2$

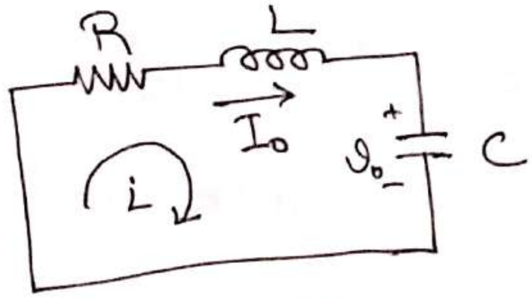
$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ A}$$

$t = 4$

$$i(5) = 2.727 + 1.273\, e^{-1.4667}$$

$$= \underline{\underline{3.02 \text{ A}}}$$

THE SOURCE-FREE SERIES RLC CIRCUIT



Consider Series RLC Circuit,

initial Capacitor voltage $\rightarrow V_0$ } at $t=0$
initial Inductor Current $\rightarrow I_0$ }

$\rightarrow$ the initial value of the derivative of i,

$$\frac{di(0)}{dt} = -\frac{1}{L}[RI_0 + V_0]$$

[This equation used to find A_2 Constant].

$\rightarrow$ Characteristic equation of i,
[roots of equation]

$$S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

where $\alpha = \frac{R}{2L}$ & $\omega_0 = \frac{1}{\sqrt{LC}}$

$\therefore$ $\boxed{S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}}$ & $\boxed{S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$

Thus, Natural response of the Series RLC circuit is,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

A_1 find by initial value $i(0)$

A_2 find by $\frac{d\,i(0)}{dt}$

Based of S_1 & S_2 there are three types of solutions

(1.) If $\alpha > \omega_0 \rightarrow$ Overdamped Case

Response is,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

(2) Critically Damped Response

$\alpha = \omega_0$

$$S_1 = S_2 = -\alpha = -\frac{R}{2L}$$

Response is,

$$i(t) = (A_2 + A_1 t)e^{-\alpha t}$$

(3) $\alpha < \omega_0$ Under damped Case

$$S_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$S_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \rightarrow$ damping frequency.

$\omega_0 \rightarrow$ Undamped frequency

Response is,

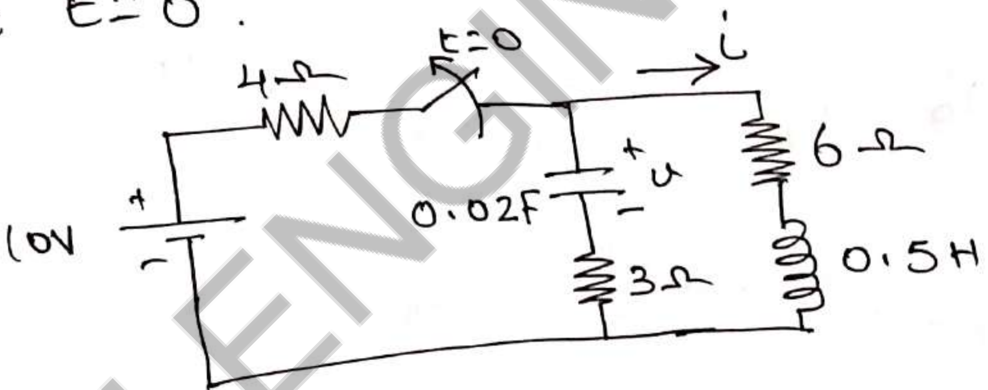
$$\boxed{i(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)}$$

time Constant $\rightarrow 1/\alpha$

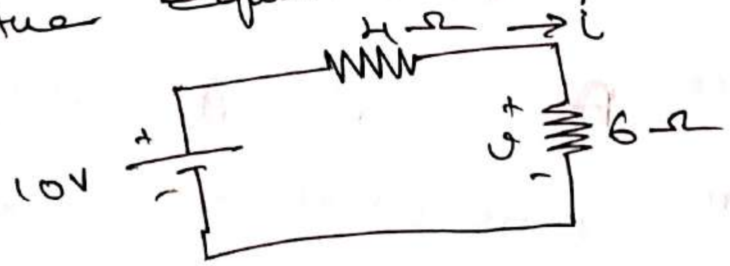
Period $\rightarrow T = 2\pi/\omega_d$

Example Problem 1:

find $i(t)$ in the Circuit. Assume that the circuit has reached Steady State at $t = 0^-$.



Solution: for $t < 0 \rightarrow$ Switch is closed. The Capacitor behaves like open circuit, & Inductor acts like Short Circuit, then the equivalent circuit is,



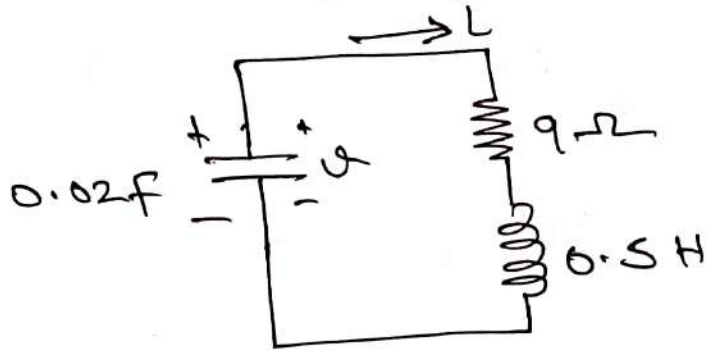
initial Current through Inductor $i(0) = \dfrac{10}{4+6} = 1\,A$ and

Voltage across ~~Capacitor~~ $v(0) = 6\,i(0) = 6 \times 1 = 6\,V$ [Voltage across 6Ω]

$i(0) \rightarrow$ Current through inductor

$v(0) \rightarrow$ Voltage across Capacitor.

* for $t > 0$, $\rightarrow$ Switch Opened, equivalent circuit is, 6Ω & 3Ω are in Series.



Source free RLC Circuit

Roots are calculated as,

$$\alpha = \frac{R}{2L} = \frac{9}{2\times 0.5} = 9$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5\times 0.02}} = 10$$

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -9 \pm \sqrt{9^2 - 10^2} = -9 \pm \sqrt{81-100}$$

$$S_{1,2} = -9 \pm j4.359 \quad [S_{1,2} = -\alpha \pm j\omega_d]$$

$\alpha < \omega_0 \rightarrow$ Underdamped

$$i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

A_1 & A_2 find by, initial conditions.

At $t=0$,

$$i(0) = 1 = A_1$$

then A_2 find by,

$$① \rightarrow \left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)]$$

$$= -2[9-6] = -6 \text{ A/s}$$

$v(0) = V_0 = -6$ → because of [opposite direction of capacitor voltage & 6 Ω resistor voltage.]

Now, derivate $i(t)$

$$② \rightarrow \frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

$$+ e^{-9t}(4.359)[-A_1 \sin 4.359t + A_2 \cos 4.359t]$$

Imposing the two equations, ① & ② at $t=0$

$$-6 = -9[A_1 + 0] + 4.359[0 + A_2]$$

$$\boxed{A_1 = 1}$$

$$-6 = -9[1] + 4.359[A_2]$$

$$A_2 = \frac{-6+9}{4.359} = \frac{3}{4.359} = 0.6882$$

Sub values A_1 & A_2 in $i(t)$

Complete response,

$$i(t) = e^{-9t}\left(\cos 4.359t + 0.6882 \sin 4.359t\right) A$$

Source Free Parallel RLC Ckt.

($\alpha > \omega_0$) Overdamped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

($\alpha = \omega_0$). Critically Damped Case

$$v(t) = (A_1 + A_2 t) e^{-\alpha t}$$

($\alpha < \omega_0$) Underdamped Case.

$$s_{1,2} = -\alpha \pm j\omega_d$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v(t) = e^{-\alpha t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

$$\frac{V_0}{R} + I_0 + C \frac{dv(0)}{dt} = 0$$

$$\frac{dv(0)}{dt} = \frac{-(V_0 + RI_0)}{RC}$$

In the parallel ckt find $v(t)$ for $t>0$, assuming $v(0) = 5V$, $i(0) = 0$, $L = 1H$ & $C = 10mF$

Consider these cases: $R = 1.923\Omega$, $R = 5\Omega$, $R = 6.25\Omega$

Case 1

if $R = 1.923\Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times 1.923\times 10\times 10^{-3}} = \underline{\underline{26}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 10\times 10^{-3}}} = \underline{\underline{10}}$$

Sin $\alpha > \omega_0$

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

& the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

Initial condition to get A_1 & A_2

$$v(0) = 5 = A_1 + A_2$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + R i(0)}{RC} = \frac{-5+0}{1.923\times 10\times 10^{-3}} = \underline{\underline{-260}}$$

By differentiating

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t=0$

$$-260 = -2A_1 - 50A_2$$

$A_1 = -0.2083$

$A_2 = 5.208$

Subs A_1 & A_2.

$$\boxed{v(t) = -0.2083e^{-2t} + 5.208\,e^{-50t}}$$

Case 2 $\quad R = 5\Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times5\times10\times10^{-3}} = 10$$

While $\omega_0 = 10$ remains the same.

$\alpha = \omega_0 = 10$, the response is critically damped,

$S_1 = S_2 = -10.$

$$v(t) = (A_1 + A_2 t)\,e^{-10t}$$

to get A_1 & A_2

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{5\times10\times10^{-3}} = -100$$

By diff.

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t=0$

$$-100 = -10A_1 + A_2$$

$A_1 = 5$

$A_2 = -50$ $\quad v(t) = (5 - 50t)\,e^{-10t}\ V$

Case 3

When $R = 6.25\,\Omega$

$$\alpha = 1/2RC = \frac{1}{2\times 6.25\times 10\times 10^{-3}} = \underline{\underline{8}}$$

While $\omega_0 = 10$ remains the same, As $\alpha < \omega_0$ in this case the response is underdamped. The roots of the characteristic eqn are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

hence $$v(t) = (A_1 \cos 6t + A_2 \sin 6t)\, e^{-8t}$$

we obtain A_1 & A_2

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = \frac{-5+0}{6.25\times 10\times 10^{-3}} = -\underline{\underline{80}}$$

But differentiating

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

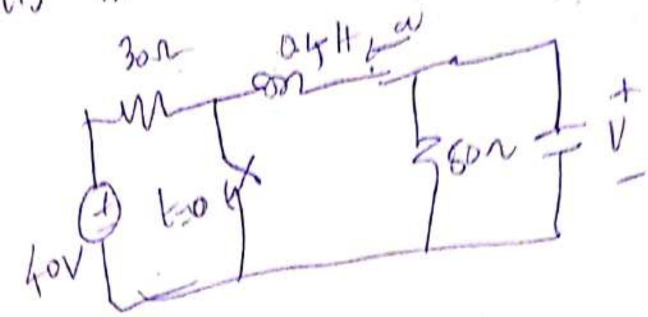
At $t=0$,

$$-80 = -8A_1 + 6A_2$$

$A_1 = 5$, $A_2 = -6.667$ Thus

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)\, e^{-8t}$$

Find v(t), for t > 0 in the RLC ckt.



Sol: t < 0

$$V(0) = \frac{50 \times 40}{50+30} = \underline{\underline{25V}}$$

initial current through the inductor

$$i(0) = -\frac{40}{30+50} = \underline{\underline{-0.5A}}$$

$$\frac{d\,v(0)}{dt} = -\frac{V(0) + R\,i(0)}{RC} = \frac{25 - 50 \times 0.5}{50 \times 20 \times 10^{-6}} = \underline{\underline{0}}$$

$\underline{\underline{t > 0}}$ Switch is closed, 30Ω & vlge. are separ.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = \underline{\underline{500}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4 \times 20 \times 10^{-6}}} = \underline{\underline{354}}$$

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250000 - 124,997.6} = -500 \pm 354$$

$$S_1 = \underline{\underline{-854}} \qquad S_2 = \underline{\underline{-146}}$$

Since $\alpha > \omega_0$, we have overdamped response

$$V(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

At $t=0$

$$V(0) = 25 = A_1 + A_2$$

$$A_2 = 25 - A_1$$

Taking the derivative of $V(t)$

$$\frac{dV}{dt} = -854 A_1 e^{-854t} - 146 A_2 e^{-146t}$$

$$\frac{dV(0)}{dt} = 0 = -854 A_1 - 146 A_2$$

$$0 = 854 A_1 + 146 A_2$$

$$A_1 = -5.156$$

$$A_2 = 30.16$$

$$V(t) = -5.156 e^{-854t} + 30.16 e^{-146t} \text{ V}$$